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Games with (Dis-)Continuous Payoff Functions and the Problem of Measurability

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Games with (Dis-)Continuous Payoff Functions and the Problem of Measurability

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Abstract. In noncooperative game theory, it is often taken for granted that expected payoffs are well-defined and independent of the integral representation. However, this need not be the case even if strategy spaces are compact and payoffs are bounded. In this paper, we establish general conditions under which the measurability requirements for working with expected payoffs are automatically met. We use our findings to enhance Glicksberg's equilibrium existence theorem and to rigorously construct the mixed extension of discontinuous games such as contests and auctions.

Keywords. Compact games, expected payoffs, measurability, Baire functions, Glicksberg's theorem, discontinuous games, u.s.c.-l.s.c. minimax theorem

JEL classification. C72: Noncooperative Games

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1 Introduction

While the *mixed extension* of a finite noncooperative game is obtained in a canonical way (von Neumann, 1928; Nash, 1950), the analogous construction for infinite games may leave some flexibility.[1] For n-person games with compact Hausdorff strategy spaces, it has been customary to have players choose regular Borel probability measures, and to define expected payoffs as iterated or joint integrals (Glicksberg, 1952; Fan, 1953). However, the game-theoretic literature has typically not discussed the conditions under which these integral representations are well-defined and consistent.

In this paper, we revisit the definition of expected payoffs in infinite n-person games in strategic form. For convenience, we focus on the case of compact strategy spaces and bounded payoff functions. Otherwise, however, the setup is general, i.e., we do not even assume that strategy spaces are Hausdorff. There are four main results.

First, to motivate the analysis, we show that expected payoffs may be undefined even under the assumptions of Mertens' (1986) u.s.c.-l.s.c. minimax theorem. To this end, we construct a two-person zero-sum game with high-cardinality strategy spaces, so that the diagonal is not measurable in the plane (see Example 1 below). Then, despite iterated expectations being well-defined, joint expectations fail to exist.[2]

Second, we derive general conditions ensuring that expected payoffs are well-defined and independent of the integral representation (Theorem 1). The key as-

[1] Cf. Aumann (1964) and Milgrom and Weber (1985). Randomization is generally useful to obtain equilibria in games that lack convex strategy spaces or quasiconcave payoff functions (Dasgupta and Maskin, 1986).

[2] In contrast to classic examples that illustrate the various ways in which the Fubini property may fail (Sierpiński, 1920a,b), our construction is not based on the Continuum Hypothesis, and therefore does not suffer from the problem of undecidability (Friedman, 1980; Fey, 2024).

sumption is that payoffs are Baire functions in the product topology. The proof first uses a corollary of the Stone-Weierstrass Theorem (Dieudonné, 1937; Stone, 1948) to show that any jointly continuous function on a product of compact topological spaces is measurable with respect to the product of the respective Borel σ-algebras, and then applies the principle of transfinite induction. Compared to standard treatments (e.g., Folland, 1999, Prop. 7.22), we drop the Hausdorff property and relax the continuity assumption on the payoff functions.

Third, we show that our sufficient conditions can be further relaxed in the case in which all strategy spaces, with possibly one exception, are second countable.[3] Specifically, Theorem 2 below assumes that payoffs are defined via a (Borel measurable and countable) case distinction over Baire functions. This result still covers the important special case where strategy spaces are metrizable and compact.

Fourth and finally, we apply our conditions to strengthen the conclusion of Glicksberg's (1952) equilibrium existence theorem (see Corollary 1). The point here is that the original work assumed the representation of expected payoffs as an iterated integral, without discussing any of the issues that are of our our concern in the present paper.[4] We also establish measurability in two examples of games with discontinuous payoff functions, illustrating how the Baire property generalizes the jointly continuous case (Example 2), and how Theorem 2 applies very naturally in an auction context (Example 3).

The rest of the paper is organized as follows. Section 2 introduces setup and notation. Section 3 discusses a motivating example. Section 4 states sufficient conditions. Section 5 deals with second countable spaces. Section 6 concerns applications. Section 7 concludes. The Appendix contains a technical lemma.

[3]A space is called *second countable* if its topology has a countable base. Any second countable space is separable, but the converse is not generally true. Cf. Kelley (1975, pp. 48–49).

[4]Perhaps for this reason, Glicksberg's equilibrium existence theorem is sometimes cited only for metrizable spaces (e.g., Fudenberg and Tirole, 1991, p. 35; Harris et al., 2005, p. 209).

2 Setup and Notation

Let $G = (X_i, u_i)_{i=1}^n$ be an n-person noncooperative game, where X_i is the set of player i's pure strategies and $u_i : X \equiv X_1 \times \ldots \times X_n \to \mathbb{R}$ is player i's payoff function, for any $i \in \{1, \ldots, n\}$. We will assume throughout that each X_i is a compact (but not necessarily Hausdorff) topological space, and that each u_i is bounded. The respective σ-algebras of *Borel sets* on X_i and X will be denoted by $\mathcal{B}(X_i)$ and $\mathcal{B}(X)$. A *Borel probability measure* μ_i on X_i is a nonnegative, countably additive set function $\mu_i : \mathcal{B}(X_i) \to [0, 1]$ such that $\mu_i(X_i) = 1$. A Borel probability measure μ_i on X_i is *regular* if for any $E \in \mathcal{B}(X_i)$ and $\varepsilon > 0$, there is a closed set $F \subseteq X_i$ and an open set $G \subseteq X_i$ such that $F \subseteq E \subseteq G$ and $\mu_i(G \setminus F) < \varepsilon$.[5] We denote by $\Delta(X_i)$ the space of all regular Borel probability measures on X_i.

In the *mixed extension* of G, each player $i \in \{1, \ldots, n\}$ chooses some $\mu_i \in \Delta(X_i)$. Given the strategy profile

$$\mu = (\mu_1, \ldots, \mu_n) \in \Delta(X_1) \times \ldots \times \Delta(X_n),$$

one may define player i's expected payoff as the iterated integral

$$E_\mu[u_i] = \int_{X_1} \left\{ \cdots \left\{ \int_{X_n} u_i(x_1, \ldots, x_n) d\mu_n(x_n) \right\} \cdots \right\} d\mu_1(x_1).$$

Alternatively, one may define expected payoffs as the joint integral

$$\widetilde{E}_\mu[u_i] = \int_{X_1 \times \ldots \times X_n} u_i(x_1, \ldots, x_n) d(\mu_1 \times \ldots \times \mu_n)(x_1, \ldots, x_n),$$

where $\mu_1 \times \ldots \times \mu_n$ denotes the product measure which, as is crucial to recall at this point, is defined on the *product σ-algebra* $\mathcal{B}(X_1) \otimes \ldots \otimes \mathcal{B}(X_n)$, rather than on the potentially larger Borel σ-algebra $\mathcal{B}(X)$.[6]

[5] Dunford and Schwartz (1958, Def. III.5.11) boils down to our definition for nonnegative Borel set functions.

[6] Cf. the Appendix.

3 Motivating Example

Notably, working with expected payoffs requires some care even if all strategy spaces are compact and payoffs are bounded.

Example 1. *Let Z be a discrete space having cardinality greater than the continuum, and let $Z^* = Z \cup \{z^*\}$ denote the Alexandroff compactification of Z.*[7] *Consider a two-person zero-sum game with $X_1 = X_2 = Z^*$, $u_1(x_1, x_2) = -u_2(x_1, x_2) = 1$ if $x_1 = x_2 \neq z^*$, and $u_1(x_1, x_2) = u_2(x_1, x_2) = 0$ otherwise. Since the topology on Z^* is Hausdorff, all Dirac measures are regular (Aliprantis and Border, 2006, p. 443). Hence, $\Delta(X_1) = \Delta(X_2) \neq \emptyset$. Moreover, for any $\mu_1 \in \Delta(X_1), \mu_2 \in \Delta(X_2)$,*

$$\begin{aligned} E_\mu[u_1] &= \int_{X_1} \left\{ \int_{X_2} u_1(x_1, x_2) d\mu_2(x_2) \right\} d\mu_1(x_1) \\ &= \int_{X_1 \setminus \{z^*\}} \mu_2(\{x_1\}) d\mu_1(x_1) \\ &= \sum_{z \in Z} \mu_1(\{z\}) \mu_2(\{z\}), \end{aligned}$$

where the sum has (at most) countably many non-zero terms and converges. Thus, iterated expectations of payoffs exist and are independent of the ordering of the players. However, the diagonal $D^* = \{(x_1, x_2) : x_1 = x_2 \in Z^*\} \subseteq X_1 \times X_2$ is not measurable with respect to $\mathcal{B}(X_1) \otimes \mathcal{B}(X_2)$ (Johnson, 1966, Ex. 4).[8] *As the singleton $\{z^*\}$ is closed in Z^*, the set $D = D^* \setminus \{(z^*, z^*)\}$ is likewise not measurable with respect to $\mathcal{B}(X_1) \otimes \mathcal{B}(X_2)$. Thus, the joint expectation of payoffs, $\widetilde{E}_\mu[u_1]$, does not exist (regardless of the choice of strategies).*

[7] Thus, $E \subseteq Z^*$ is open if and only if either $E \subseteq Z$ or E is cofinite (or both).

[8] Here is a sketch of the proof (cf. Halmos, 1974, p. 261). To provoke a contradiction, suppose that $D^* \in \mathcal{B}(Z^*) \otimes \mathcal{B}(Z^*)$. Then, there exists a countable class $\mathbf{R}$ of rectangles such that D^* is an element of $\Sigma_{\mathbf{R}}$, the σ-algebra generated by $\mathbf{R}$. Let $\mathbf{E}$ and $\Sigma_{\mathbf{E}}$, respectively, denote the class of sides of rectangles belonging to $\mathbf{R}$ and the σ-algebra generated by $\mathbf{E}$. Since $D^* \in \Sigma_{\mathbf{E}} \otimes \Sigma_{\mathbf{E}}$, every section of D^* belongs to $\Sigma_{\mathbf{E}}$. Thus, $\Sigma_{\mathbf{E}}$ has cardinality greater than the continuum. However, this is impossible because $\mathbf{E}$ is countable.

If, in the example, $x_2 \in X_2$ is fixed, then $u_1(\cdot, x_2)$ is either constant or the indicator of a closed singleton. Similarly, if $x_1 \in X_1$ is fixed, then $u_2(x_1, \cdot)$ is either constant or the negative indicator of an open singleton. In both cases, any upper contour set is closed. Thus, the game is *u.s.c.-l.s.c.* in the sense of Mertens (1986).

4 Sufficient Conditions

Let $\mathcal{C}(X)$ denote the space of jointly continuous functions on $X = X_1 \times \ldots \times X_n$. A hierarchy of increasingly wider classes of functions, known as *Baire functions of class* α, where α is a countable ordinal, is defined by transfinite induction as follows.[9]

Definition 1. $\mathfrak{B}_0(X) = \mathcal{C}(X)$ *is the set of all continuous functions* $f\colon X \to \mathbb{R}$*. For any countable ordinal* $\alpha > 0$*, a function* $f\colon X \to \mathbb{R}$ *belongs to* $\mathfrak{B}_\alpha(X)$ *if there exists a sequence* $\{f_k\}_{k=1}^{\infty}$ *with each* $f_k \in \bigcup_{\alpha'<\alpha} \mathfrak{B}_{\alpha'}(X)$ *such that* $f(x) = \lim_{k\to\infty} f_k(x)$*, for all* $x \in X$*. A function* $f\colon X \to \mathbb{R}$ *is called a* ***Baire function*** *if* $f \in \mathfrak{B}_\alpha(X)$ *for some countable ordinal* α*.*

The class of Baire functions is, consequently, the smallest class of functions that contains all jointly continuous functions and is closed under taking pointwise limits of arbitrary sequences.

The following result shows that all goes well if payoff functions are of the Baire type.

Theorem 1. *Suppose that the pure strategy spaces* $X_1, \ldots, X_n$ *are compact, and that player* i*'s payoff function* u_i *is bounded and of the Baire type. Then, the iterated expectation* $E_\mu[u_i]$ *and the joint expectation* $\widetilde{E}_\mu[u_i]$ *are well-defined and coincide,*

[9] For background on the principle of transfinite induction and countable ordinals, see Folland (1999, pp. 9–10). A non-technical introduction to Baire functions is Lorch (1971). See also Kechris (1995).

i.e., $E_\mu[u_i] = \widetilde{E}_\mu[u_i]$*. In particular, the iterated expectation does not depend on the order of integration.*

Proof. The proof has three steps.

Step 1. We show first that, if u_i is jointly continuous, then u_i is measurable with respect to $\mathcal{B}(X_1) \otimes \ldots \otimes \mathcal{B}(X_n)$. Consider the set $\mathfrak{A} \subseteq \mathcal{C}(X)$ consisting of all finite sums of functions of the form

$$(x_1, \ldots, x_n) \mapsto \varphi_1(x_1) \cdot \ldots \cdot \varphi_n(x_n),$$

where $\varphi_i : X_i \to \mathbb{R}$ is continuous for each $i \in \{1, \ldots, n\}$. According to Stone (1948, Thm. 14), any $u_i \in \mathcal{C}(X)$ is the uniform limit of a sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathfrak{A}$. Since the canonical projection $\pi_i : X \to X_i$ is, by definition, measurable with respect to $\mathcal{B}(X_1) \otimes \ldots \otimes \mathcal{B}(X_n)$, and finite sums and products of measurable functions are measurable (Folland, 1999, Prop. 2.6), each f_k is measurable with respect to the product σ-algebra. As even the pointwise limit of a sequence of measurable functions is measurable (Folland, 1999, Prop. 2.7), u_i is seen to be measurable with respect to $\mathcal{B}(X_1) \otimes \ldots \otimes \mathcal{B}(X_n)$.

Step 2. Suppose next that u_i is a Baire function. Then, there exists a countable ordinal α such that $u_i \in \mathfrak{B}_\alpha(X)$. If $\alpha = 0$, then u_i is continuous, and we know from the previous step that u_i is measurable with respect to the product σ-algebra. If, however, $\alpha > 0$, then straightforward transfinite induction shows that u_i is measurable with respect to $\mathcal{B}(X_1) \otimes \ldots \otimes \mathcal{B}(X_n)$.

Step 3. By assumption, u_i is bounded. Hence, given the measurability property established in the previous step, Fubini's theorem (Dunford and Schwartz, 1958, Thm. III.11.9) implies that $E_\mu[u_i]$ and $\widetilde{E}_\mu[u_i]$ are well-defined and equal to each other. This concludes the proof. □

Note that the assumptions of Theorem 1 fail in Example 1. Specifically, the payoff functions in that example are not of the Baire type. If they were, they would be measurable with respect to the product σ-algebra, as shown in the proof, but they are not. In particular, the assumption that payoffs are of the Baire type cannot be dropped without losing the conclusion of the theorem.

5 Second Countable Spaces

When all pure strategy spaces are second countable, possibly with one exception, then the product of the respective Borel σ-algebras equals the Borel σ-algebra of the joint strategy space. This was shown by Johnson (1966) for two Hausdorff factors, but his argument extends to any finite number of possibly non-Hausdorff factors, as we explain in the Appendix. Using this fact, the Baire class condition on the payoff functions can be substantially relaxed in the case of second countable strategy spaces. We propose the following definition.

Definition 2. *A function g on X is **piecewise Baire** if there exists a partition $X = \cup_{m=1}^{M} B_m$ into Borel sets $B_m \in \mathcal{B}(X)$ and functions of the Baire type $\{g_m\}_{m=1}^{M}$ on X, for some $1 \leq M \leq \infty$, such that $g(x) = g_m(x)$ if $x \in B_m$.*

Thus, a function is piecewise Baire if it results from countably many Baire functions via a Borel measurable case distinction. This includes, of course, the case of a finite case distinction. We arrive at the announced variant of Theorem 1.

Theorem 2. *Suppose that the pure strategy spaces $X_1, \ldots, X_n$ are compact and (with possibly one exception) second countable, and that player i's payoff function u_i is bounded and piecewise Baire. Then, the conclusion of Theorem 1 holds.*

Proof. By Lemma A.1 in the Appendix, the indicator function $\mathbf{1}_{x \in B_m}$ is measurable with respect to the σ-algebra $\mathcal{B}(X_1) \otimes \ldots \otimes \mathcal{B}(X_n)$. This is, therefore, also the case for any finite sum $g^{(M')}(x) = \sum_{m=1}^{M'} g_m(x)\mathbf{1}_{x \in B_m}$, where $1 \leq M' \leq M$ with $M' < \infty$. Clearly, $\lim_{M' \to M} g^{(M')}(x) = u_i(x)$ for any $x \in X$. Thus, u_i is measurable with respect to the product σ-algebra, and the claim follows as before. □

Thus, under the additional assumption of second countability for the pure strategy spaces of all but at most one player, countable case distinctions with respect to a Borel measurable partition are admissible. Notably, Theorem 2 applies when pure strategy spaces are compact and metrizable.

The assumption of second countability cannot be dropped from the statement of Theorem 2. Indeed, in Example 1, the diagonal D^* is closed in $X = X_1 \times X_2$ as a consequence of the Hausdorff property of the topology on Z^*, so that D is measurable with respect to $\mathcal{B}(X)$. Therefore, payoffs are piecewise Baire. However, the topology on Z^* is not second countable, and the conclusion of the theorem fails to hold in the example.

6 Applications

In this section, we illustrate the application of Theorems 1 and 2. Specifically, we extend Glicksberg's theorem (Subsection 6.1) and discuss two examples of games with discontinuous payoff functions (Subsection 6.2).

6.1 Glicksberg's Theorem

Glicksberg (1952) defined expected payoffs as an iterated expectation.[10] Theorem 1 shows that, for jointly continuous functions, expected payoffs are indeed well-defined and independent of the integral representation. We therefore obtain the following extension of Glicksberg's equilibrium existence result.

[10]The notation used by Glicksberg (1952, p. 172) is consistent with Halmos (1974, p. 146).

Corollary 1. *Suppose that strategy spaces are compact and Hausdorff, and that payoff functions are continuous. Then, there exists a mixed-strategy Nash equilibrium even if expected payoffs are defined as joint expectations.*

Thus, with respect to Glicksberg's (1952) existence theorem for mixed-strategy Nash equilibria, we conclude that his definition of expected payoffs is rigorous under the general assumptions of his analysis, and that the way in which he chose to define expected payoffs (i.e., as an iterated expectation) is equivalent to the alternative definition as a joint expectation.

6.2 Discontinuous Games

The following examples illustrate the application of our conditions to discontinuous games.

Example 2 (**Rent-Seeking Contest**). *In a rent-seeking contest with parameter* $R > 0$ *(Tullock, 1980),* $n \geq 2$ *players each choose an effort* $x_i \in X_i \equiv [0, x_i^{\max}]$, *for some* $x_i^{\max} > 0$, *and payoffs are given by* $u_i(x_1, \ldots, x_n) = \left(x_i^R / \sum_{j=1}^n x_j^R \right) - x_i$ *if* $\max_{j \in \{1,\ldots,n\}} x_j > 0$, *and by* $u_i(x_1, \ldots, x_n) = s_i$ *otherwise, where* $s_1, \ldots, s_n \geq 0$ *are constants such that* $\sum_{j=1}^n s_j = 1$. *Since* $u_i(x_1, \ldots, x_n)$ *is the pointwise limit of* $f_k(x_1, \ldots, x_n) = \left((x_i^R + \frac{s_i}{k}) / \sum_{j=1}^n (x_j^R + \frac{s_j}{k}) \right) - x_i$ *for* $k \to \infty$, *the game is of Baire class one. By Theorem 1, expected payoffs are well-defined and independent of the integral representation.*

Example 3 (**All-Pay Auction**). *In the* n*-bidder all-pay auction with valuations* $v_1, \ldots, v_n > 0$ *(Baye et al., 1996), players each choose a bid* $x_i \in X_i \equiv [0, v_i]$, *and payoffs are given by* $u_i(x_1, \ldots, x_n) = \frac{v_i}{m} - x_i$ *if* $x_i = \max_{j \in \{1,\ldots,n\}} x_j$ *and* $m = \# \arg\max_{j \in \{1,\ldots,n\}} x_j$, *and by* $u_i(x_1, \ldots, x_n) = -x_i$ *if* $x_i < \max_{j \in \{1,\ldots,n\}} x_j$. *Since* $u_i(x_1, \ldots, x_n)$ *is defined through a* $\mathcal{B}(X)$*-measurable case distinction involv-*

ing continuous functions, Theorem 2 implies that expected payoffs are well-defined and independent of the integral representation.

More generally, our conditions should be valuable when applying the main results for mixed-strategy equilibrium existence, such as Reny (1999), Monteiro and Page (2007), and Prokopovych and Yannelis (2014), for instance.

7 Conclusion

In conclusion, we hope that our analysis creates a broader awareness of the assumptions implicitly made when working with the mixed extension of a game with infinite pure strategy spaces. At the same time, given the sufficient conditions derived above, we feel that it should not be too demanding to check measurability in specific examples and to thereby ensure that the game-theoretic analysis is rigorous also in this respect. Especially the punchline from Theorem 2 that measurability assumptions needed to safely work with expected payoffs are automatically satisfied if (i) pure strategy spaces are compact and second countable and (ii) payoff functions are bounded and piecewise Baire should prove useful in a wide range of game-theoretic applications.

A Appendix

The following lemma is used in the proof of Theorem 2.

Lemma A.1 (**Johnson, 1966**). *Let* $X = X_1 \times \ldots \times X_n$ *be a product of topological spaces. Then,* $\mathcal{B}(X_1) \otimes \ldots \otimes \mathcal{B}(X_n) \subseteq \mathcal{B}(X)$. *If* X_i *is second countable, for* $i \in \{1, \ldots, n-1\}$, *then* $\mathcal{B}(X_1) \otimes \ldots \otimes \mathcal{B}(X_n) = \mathcal{B}(X)$.[11]

[11] In general, the inclusion may be strict, as Example 1 illustrates.

Proof. Let $\Sigma_i = \{B_i \subseteq X_i : \pi_i^{-1}(B_i) \in \mathcal{B}(X)\}$. By continuity, any open set $B_i \subseteq X_i$ is an element of Σ_i. Hence, noting that Σ_i is a σ-algebra, $\mathcal{B}(X_i) \subseteq \Sigma_i$. Thus, each π_i is measurable with respect to $\mathcal{B}(X)$, proving the first claim. Next, suppose that $n-1$ factors are second countable, and let $U \subseteq X$ be open. For any $x = (x_1, \ldots, x_n) \in U$, there are open sets $U_1, \ldots, U_n$ such that $x \in U_1 \times \ldots \times U_n$. Without loss of generality, $U_1, \ldots, U_{n-1}$ are base sets, and $U_n = \bigcup\{V_n \subseteq X_n \text{ open} : U_1 \times \ldots \times U_{n-1} \times V_n \subseteq U\}$. Thus, U is a countable union of products of open sets, proving also the second claim. □

Conflict of Interest Statement

There are no conflicts of interest to declare.

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List of Notation (not for Publication)

Symbol	Interpretation
$G = (X_i, u_i)_{i=1}^n$	n-person game
$X = X_1 \times \ldots \times X_n$	space of pure strategy profiles
$\mathcal{B}(X_i), \mathcal{B}(X)$	Borel σ-algebras on X_i and X, respectively
μ_i	Borel probability measure on X_i
E, F, G	sets used to define regularity
$\Delta(X_i)$	space of all regular Borel probability measures
$\mu = (\mu_1, \ldots, \mu_n)$	mixed strategy profile
$\mu_1 \times \ldots \times \mu_n$	product measure
$E_\mu[u_i], \widetilde{E}_\mu[u_i]$	i's expected payoff as iterated and joint integral
$\mathcal{B}(X_1) \otimes \ldots \otimes \mathcal{B}(X_n)$	product of Borel σ-algebras
Z	discrete space with cardinality greater than $\mathfrak{c}$
$Z^* = Z \cup \{z^*\}$	one-point compactification
D, D^*	diagonals in Z and Z^*
$\mathbf{R}, \mathbf{E}, \Sigma_{\mathbf{R}}, \Sigma_{\mathbf{E}}$	sets of rectangles and sides, σ-algebras
α, α'	countable ordinals
$\mathcal{C}(X)$	space of all continuous functions on X
$\mathfrak{B}_\alpha(X)$	Baire classes on X
f, g	functions on X
$\{f_k\}_{k=1}^\infty$	sequence of functions on X
$\mathfrak{A}$	Stone's algebra
$\varphi_i(x_i)$	function on X_i
$\pi : X \to X_i$	canonical projection
$X = \bigcup_{m=1}^M B_m$	Borel partition of X, with $1 \leq M \leq \infty$
g_m	function on X, where $1 \leq m \leq M$
$g^{(M')}(x) = \sum_{m=1}^{M'} g_m(x)$	partial sum, where $1 \leq M' \leq M$ and $M' < \infty$
$x_i^{\max}$	i's maximum effort
R	Tullock's parameter
$s_1, \ldots, s_n \in [0, 1]$	constants such that $\sum_{j=1}^n s_j = 1$
v_i	player i's valuation
U, U_i, V_n	open sets in X, X_i, and X_n
B_i	subset of X_i
Σ_i	σ-algebra on X_i